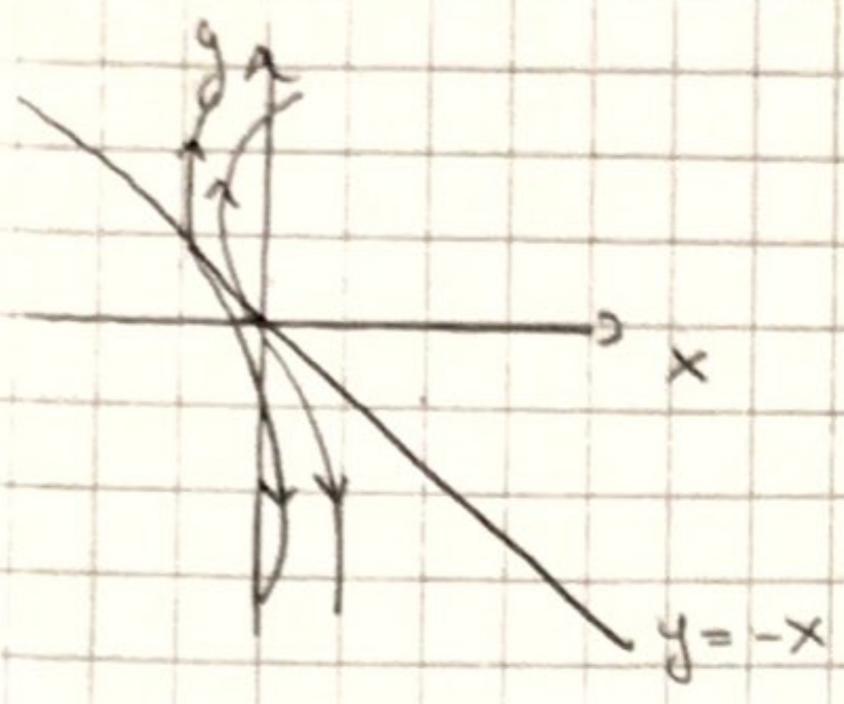
$(A-2E)\begin{pmatrix}h_{11}\\ h_{12}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$h_1=\begin{pmatrix}1\\ -1\end{pmatrix}$ $\quad y=-x$



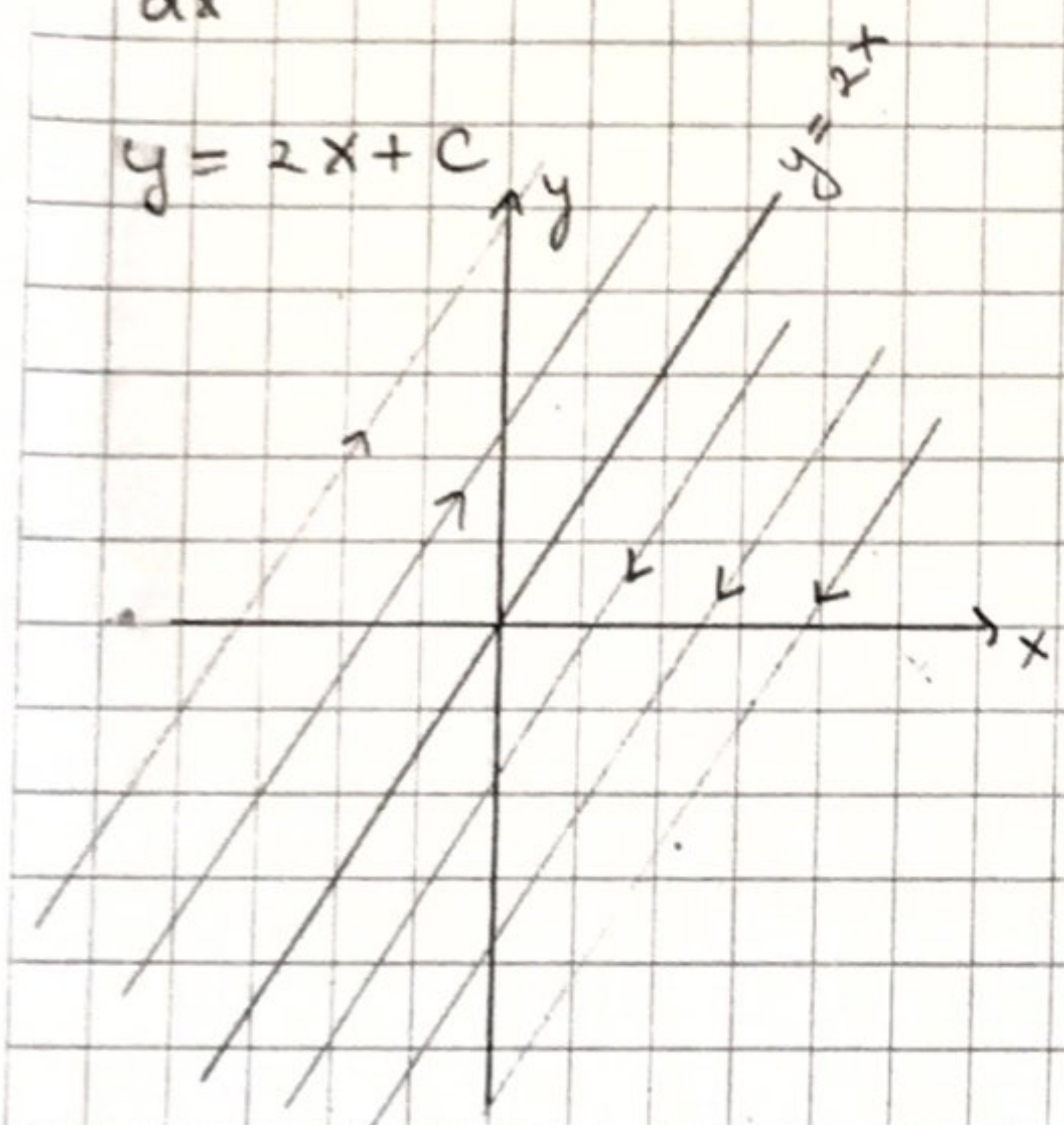
8. $x' = y - 2x$

$y' = 2y - 4x$

$\lambda_1 = \lambda_2 = 0$ prave (cijela fazna ravan)

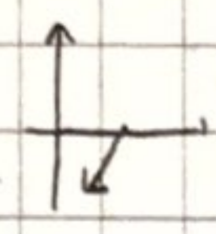
$\frac{dy}{dx} = 2$

$y = 2x + C$



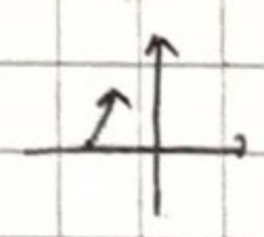
(1,0) proizvoljno $\quad x' = -2$ ↓

$y' = -4$ ↓



(-1,0) proizvoljno $\quad x' = 2$ ↗

$y' = 4$ ↗



Sistemi koji nisu linearni i centar im nije u (0,0)

9. $x_1' = \sqrt{x_1^2 - x_2 + 2} - 2$

$x_2' = \operatorname{arctg}(x_1^2 + x_1 x_2)$

$x_1^2 - x_2 + 2 = 4 \Rightarrow x_2 = x_1^2 - 2 \quad (1)$

$x_1^2 + x_1 x_2 = 0 \quad (2)$

$(1) \to (2) \quad x_1^2 + x_1(x_1^2 - 2) = 0$

$x_1(x_1 + x_1^2 - 2) = 0$

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$x_1 = 0 \qquad -2 = x_2$

$x_1 = 1 \qquad -1 = x_2$ položaji ravnoteže

$x_1 = -2 \qquad 2 = x_2$

za tačku $(0,-2)$:

$y_1 = x_1$ ← nujena

$y_2 = x_2 + 2 \qquad x_2 = y_2 + 2$

$x_1' = y_1'$

$y_2' = x_2'$

$$y_1' = \sqrt{y_1^2 - (y_2 - 2) + 2} - 2 = \sqrt{y_1^2 - y_2 + 4} - 2$$

$$y_2' = \operatorname{arctg}(y_1^2 + y_1 y_2 - 2y_1)$$

Sad za ovaj sistem ispitujemo položaj u tački $(0,0)$ a to je ekvivalentno sa položajem u tački $(0,-2)$.

$$y_1' = \left(\frac{2y_1}{2\sqrt{y_1^2 - y_2 + 4}}\right)_{(0,0)} \cdot y_1 + \left(\frac{-1}{2\sqrt{-\,\cdot\,-}}\right)_{(0,0)} y_2$$

$y_2' = \ldots$

$$\left.\begin{aligned} y_1' &= 0 \cdot y_1 - \frac{1}{4} y_2 \\ y_2' &= -2 y_1 + 0 \cdot y_2 \end{aligned}\right\} \Rightarrow \lambda_{1,2} = \pm \frac{1}{\sqrt{2}} \quad \text{sedlo}$$

$x' = f_1(x, y)$

$y' = f_2(x, y)$

sedlo, fokus i čvorovi se čuvaju preko grubi položaj

Onda ćemo ga uvesti na lin. sistem:

$x' = D_1 f_1(0,0) \cdot x + D_2 f_1(0,0)\, y$

$y' = D_1 f_2(0,0) \cdot x + D_2 f_2(0,0) \cdot y$

... $(-2, 2)$:

$y_1 = x_1 + 2$ $\leftarrow$ smjena

$y_2 = x_2 - 2$

$y_1' = \sqrt{(y_1-2)^2 - y_2} - 2$

$y_2' = \operatorname{arctg}\left((y_1-2)^2 + (y_1-2)(y_2+2)\right)$

$y_1' = \dots$

$y_2' = \dots$

$$\left.\begin{aligned} y_1' &= -y_1 - \frac{1}{4}y_2 \\ y_2' &= -2y_1 - 2y_2 \end{aligned}\right\} \Rightarrow \lambda_{1,2} = \frac{-3 \pm \sqrt{3}}{2}$$ - stabilni čvor

Posljedično, $(-2, 2)$ je stabilni čvor za naš sistem.

... tačku $(1, -1)$:

$y_1 = x_1 - 1$

$y_2 = x_2 + 1$

$$\left.\begin{aligned} y_1' &= \frac{y_1}{2} - \frac{y_2}{4} \\ y_2' &= y_1 + y_2 \end{aligned}\right\} \Rightarrow \lambda_{1,2} = \frac{3}{4} \pm \frac{i\sqrt{3}}{4}$$ nestabilni fokus

V 18.05.2018.

$$\left.\begin{aligned} x_1' &= f_1(x_1, x_2) \\ x_2' &= f_2(x_1, x_2) \end{aligned}\right\} (1)$$

$f_1, f_2 \in C^2(\mathbb{R}^2)$

$f_1(x_1, x_2) = D_1 f_1(0,0) \cdot x_1 + D_2 f_1(0,0) \cdot x_2 + R_1(x_1, x_2)$

$f_2(x_1, x_2) = D_1 f_2(0,0) \cdot x_1 + D_2 f_2(0,0) \cdot x_2 + R_2(x_1, x_2)$

$f_1(0,0) = f_2(0,0) = 0$
(da bi bio položaj ravnoteže)

Ako $\exists \alpha > 0$ t.d.

$$\frac{R_1(x_1, x_2)}{\|(x_1, x_2)\|^{1+\alpha}} \longrightarrow 0$$

$$\frac{R_2(x_1, x_2)}{\|(x_1, x_2)\|^{1+\alpha}} \longrightarrow 0$$

tada je sing. tačka $(0,0)$ sistema (1) istog tipa kao i sing. tačka $(0,0)$ sistema $x_1' = D_1 f_1(0,0) \cdot x_1 + D_2 f_1(0,0) \cdot x_2$

$x_2' = D_1 f_2(0,0) \cdot x_1 + D_2 f_2(0,0) \cdot x_2$

u svim slučajevima izuzev kad je centar.

(1) $x_1' = x_1(x_1 + x_2 - 2)$

$x_2' = x_2(1 - x_1)$

$x_2(1 - x_1) = 0 \Rightarrow$ Tačke su: $(0,0), (2,0), (1,1)$

$x_2 = 0 \qquad x_1 = 1$

$x_1 = 0 \qquad x_2 = 1$

$\vee$

$x_1 = 2$

$(1,1)$

$y_1 = x_1 - 1 \qquad x_1 = y_1 + 1$

$y_2 = x_2 - 1 \qquad x_2 = y_2 + 1$

$x_1' = y_1'$

$x_2' = y_2'$

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$y_1' = (y_1+1)(y_1+y_2)$

$y_2' = (y_2+1)(-y_1)$

Sad je tačka $(0,0)$ položaj ovog sistema.

$D_1 f_1 = y_1 + y_2 + y_1 + 1 \xrightarrow{(0,0)} 1$ - po prvoj prouj. izvod

$D_2 f_1 = y_1 + 1 \xrightarrow{(0,0)} 1$ - po drugoj prouj. izvod

$D_1 f_2 = -(y_2+1) \xrightarrow{(0,0)} -1$

$D_2 f_2 = -y_1 \xrightarrow{(0,0)} 0$

$y_1' = 1 \cdot y_1 + 1 \cdot y_2$

$y_2' = -y_1 \cancel{+0 \cdot y_2}$

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$\det(A-\lambda E) = (1-\lambda)(-\lambda)+1 = \lambda^2 - \lambda + 1 \Rightarrow \lambda_{1,2} = \frac{1 \pm \sqrt{3}i}{2}$ nestabilan fokus

⊙ Fokus se čuva preko teoreme.

$(0,0)$

Ne moramo da vršimo translaciju.

$D_1 f_1 = x_1 + x_2 - 2 + x_1 \xrightarrow{(0,0)} -2$

$D_2 f_1 = x_1 \longrightarrow 0$

$D_1 f_2 = -x_2 \longrightarrow 0$

$D_2 f_2 = 1 - x_1 \longrightarrow 1$

$$A = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

$\lambda_1 = -2$

sedlo

$\lambda_2 = 1$

(2,0)

$x_1 = y_1 + 2$

$x_2 = y_2$

$y_1' = (y_1+2)(y_1+y_2)$

$y_2' = y_2(-y_1-1)$

$\Rightarrow (0,0)$ tačka položaja ovog sis

⋮

$A = \begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix}$

$\lambda_1 = -1$

sedlo

$\lambda_2 = 2$

Za vježbu: $x_1' = \ln(2 - x_2^2)$

$x_2' = e^{x_1} - e^{x_2}$

Tačke $(1,1)$ ... $\rightarrow$ stabilni fokus

$(-1,-1)$ ... $\rightarrow$ sedlo

Stabilnost rešenja po Ljapunovu

- proučava neprekidnu zavisnost rešenja od početnih usl. na beskonačnom int.

Posm. d.j. $x' = f(t,x)$ (1)

Def. Rešenje $x = \varphi(t)$, $\varphi(t_0) = \varphi_0$ d.j. (1) je stabilno po Ljapunovu kad $t \to \infty$ ako $(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon) > 0)$ t.d. $\forall x = x(t)$, $x(t_0) = x_0$ rješ. jedn.(1) iz $|x_0 - \varphi_0| < \delta \Rightarrow |x(t) - \varphi(t)| < \varepsilon$ $\forall t \in [t_0, +\infty)$.

- stabilnost rešenja po Ljapunovu znači da bliskost početnih uslova uslovljava bliskost rešenja za $\forall t \geq t_0$

Def. Rešenje $x = \varphi(t)$, $\varphi(t_0) = \varphi_0$ d.j.(1) $x' = f(t,x)$ je asimptotski stabilno po Ljapunovu kad $t \to \infty$ ako je stabilno i ako $\exists \sigma > 0$ t.d. $\forall x = x(t)$, $x(t_0) = x_0$ ove jedn. $|x_0 - \varphi_0| < \sigma \Rightarrow \lim_{t \to \infty} |x(t) - \varphi(t)| = 0$.

rešenje $t \geq t_0$

za sistem

Def. $x' = F(t, x)$ $\qquad \varphi(t)$ rj.

$x = \phi(t)$ $\qquad \varphi(t_0) = \varphi_0$ $\qquad x(t_0)$ //

nazivamo <u>stabilnim po Ljapunovu</u> ako $\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \ \forall x_0 \ \|x_0 - \varphi_0\| < \delta \Rightarrow$

$\forall t \geq t_0 \ \|x(t) - \varphi(t)\| < \varepsilon$.

(Analogno za asimptotsku stabilnost)

2) Ispitati stabilnost po Ljapunovu:

$x' = x(4 - t^2)$

$x(0) = 0$

$$\frac{dx}{x} = (4 - t^2)\,dt \quad , x \neq 0$$

$$\ln(x) = 4t - \frac{t^3}{3} + \ln(c)$$

$$\left.\begin{array}{l} x = c \cdot e^{4t - \frac{t^3}{3}} \\ x(0) = 0 \end{array}\right\} \Rightarrow \varphi \equiv 0$$

Ispitujemo u odnosu na tačku $x(0) = x_0$

$$\Rightarrow x(t) = x_0 \cdot e^{4t - \frac{t^3}{3}}$$

$|x_0 - 0| < \delta$

$\varepsilon > 0$

Treba naći δ t.d. $|x(t) - \underset{0}{\varphi(t)}| < \varepsilon$

$$|x(t)| < \varepsilon$$

$$\Rightarrow |x_0| \cdot e^{4t - \frac{t^3}{3}} < \varepsilon$$

$t \geq 0$

Gledamo max f-je: $\max\limits_{t \geq 0} \left(4t - \frac{t^3}{3}\right) = \frac{16}{3}$

$$h(t) = 4t - \frac{t^3}{3}$$

$$h'(t) = 4 - t^2$$

	$-\infty$	-2	0	2	$+\infty$
h'			+	−	
			↗	↘	

$\Rightarrow 2$ je max vr. $\qquad h(2) = 8 - \frac{8}{3} = \frac{16}{3}$

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$M = e^{\frac{16}{3}}$

$\Rightarrow |x(t)| = |x_0| \, e^{4t - \frac{t^3}{3}} < \varepsilon$

$\underline{\underline{\varepsilon > 0}}$, onda ćemo δ birati t.d. $\delta = \frac{\varepsilon}{M}$ jer

$$|x(t)| \leq |x_0| \cdot M < \frac{\varepsilon}{\cancel{M}} \cdot \cancel{M} = \varepsilon$$

$\Rightarrow$ Jeste stabilno.

Štaviše asimptotski stabilno jer $\lim\limits_{t \to +\infty} x(t) = 0$

jer $e^{4t - \frac{t^3}{3}} \xrightarrow{t \to +\infty} e^{-\infty} = 0$

(3.) $3(t-1)x' = x$, $t \geq 1$ DJ koja RP

$x(2) = 0$

$\underline{\underline{2}}$

$$\frac{dx}{x} = \frac{1}{3} \frac{dt}{t-1} \Big/ \int$$

$$\ln(x) = \ln\left((t-1)^{\frac{1}{3}}\right) + \ln(c)$$

$$x(t) = c \cdot (t-1)^{\frac{1}{3}}$$

$x(2) = 0$

$0 = c \cdot 1 \Rightarrow c = 0$

$\boxed{x \equiv 0}$

$x(2) = x_0$

$x(2) = c$

$\boxed{x(t) = x_0 (t-1)^{\frac{1}{3}}}$

$\rightarrow$ upoređujemo ova 2 reš.

$\exists \varepsilon > 0 \ \forall \delta \ \exists x(t)$
$x(t_0) = x_0$
$|x_0 - y_0| < \delta \ \wedge \ |x(t^*) - y(t^*)| \geq \varepsilon$
$t^* > t_0$

Pokazaćemo da nije stabilno.

$\varepsilon = 1$

$|x_0| < \delta$

očekivano je jer $t \to +\infty$

$$|x(t)| = |x_0|(t-1)^{\frac{1}{3}} > 1 \text{ za } (t-1)^{\frac{1}{3}} > \frac{1}{|x_0|}$$

$$t - 1 > \frac{1}{|x_0|^3} \Rightarrow t > 1 + \frac{1}{|x_0|^3} \text{ nestab.}$$

(4.) $x' + x = 1 + t$ linearna DJ

$x(0) = 0$

$x = e^{\int -dt}\left(\int (1+t) e^{\int dt} + c\right) = e^{-t}\left(\int (1+t)e^{t} dt + c\right) =$

$= e^{-t}(e^{t} \cdot t + c) = t + c \cdot e^{-t}$

$x(0) = 0$, $c = ?$

$0 = 0 + c \cdot 1 \Rightarrow c = 0$

$\boxed{e(t) = t}$

$x(0) = x_0$

$x_0 = 0 + c \cdot 1$

$\Big\} x_0 = c$

$\boxed{x(t) = t + x_0 \cdot e^{-t}}$

$|x(t) - e(t)| = |x_0| \cdot \underbrace{e^{-t}}_{\leq 1} \leq |x_0|$

$\delta = \varepsilon$

$\varepsilon > 0$

$|x_0 - 0| < \delta$

$|x_0| < \delta$

| asimptotski je stabilna, $\lim_{t \to +\infty} e^{-t} = 0$

Ovaj primjer pokazuje da postoje f-je koje nisu ograničene ali su stabilne. Postoji i suprotno, ograničene i nestabilne.

(5.) Ispitati stab. triv. rješ; $t = 0$.

$x_1' = -x_1$

$x_2' = t x_2$

$\Rightarrow x_1 = c_1 \cdot e^{-t}$

$\Rightarrow x_2 = c_2 \cdot e^{t}$

$x_1(0) = a \Rightarrow c_1 = a \Rightarrow x_1(t) = a\cdot e^{-t}$

$x_2(0) = b \Rightarrow c_2 = b \Rightarrow x_2(t) = b\cdot e^{kt}$

$\|(a,b)-(0,0)\| < \delta$

$\| x(t) - e(t) \| = \sqrt{a^2 e^{-2t} + b^2 e^{2kt}}$

ako je $k \le 0$, $t \ge 0$ $\Rightarrow \le \sqrt{a^2+b^2} < \delta = \varepsilon$

$x_0 = \begin{pmatrix} a \\ b \end{pmatrix}$ $\qquad \|x_0 - e_0\| < \delta$

$e_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad \sqrt{a^2+b^2} < \delta$

- stabilno
- asimptotski stab. $\left\{ \begin{array}{l} k<0 \text{ jeste as. stabilno} \\ k=0 \text{ nije } -\,\text{//}\,- \end{array} \right.$

Ali, ako je $k > 0 \Rightarrow \lim\limits_{t\to+\infty} \|x(t) - e(t)\| = +\infty$ pa ovo ne možemo da napravimo proizvoljno malim.

⑥ Ispitati stabilnost trivijalnog rešenja sistema DJ-a ako je poznato njihovo opšte rešenje. ; $t=0$.

$x_1(t) = c_1 \cos^2 t - c_2 e^{-t}$

$x_2(t) = c_1 t^4 e^{-t} + 2c_2$

$\phi : \ \phi' = A\cdot\phi$

početni uslovi $\left\{ \begin{array}{l} x_1(0) = a \\ x_2(0) = b \end{array} \right.$

$a = c_1 \cdot 1 - c_2 \cdot 1$

$b = 2c_2$

$c_2 = \frac{b}{2}$

$c_1 = a + \frac{b}{2}$

$\varphi_1(0) = 0$

$\varphi_2(0) = 0$

$x_1(0) = a$

$x_2(0) = b$

J-ne kri. $\quad x_1(t) = \left(a + \frac{b}{2}\right)\cos^2 t - \frac{b}{2} e^{-t}$

$x_2(t) = \left(a + \frac{b}{2}\right) t^4 e^{-t} + b$

$$\max_{t \geq 0} (t^4 \cdot e^{-t}) = M$$

$$h(t) = t^4 \cdot e^{-t}$$

$$h'(t) = 4t^3 e^{-t} - t^4 e^{-t} = e^{-t} \cdot t^3 (4-t)$$

	0		4		$+\infty$
h'		+		−	
		$\nearrow$		$\searrow$	

$\Rightarrow$ u 4 je max

$$\| x(t) - \underset{0}{\underset{\parallel}{e(t)}} \| = \sqrt{x_1^2(t) + x_2^2(t)}$$

$$(a \pm b)^2 \leq 2(a^2 + b^2)$$

$$a^2 \pm 2ab + b^2 \leq 2(a^2+b^2)$$

$$\pm 2ab \leq a^2 + b^2$$

$$\Downarrow$$

$$\leq \sqrt{2\left[(a+\frac{b}{2})^2 \cos^4 t + \frac{b^2}{4} e^{-2t}\right] + 2\left[(a+\frac{b}{2})^2 (t^4 e^{-t})^2 + b^2\right]}$$

$$\cos^4 t \leq 1$$

$$e^{-2t} \leq 1$$

$$(t^4 e^{-t})^2 \leq M^2$$

$$\leq \sqrt{2(a+\frac{b}{2})^2 + \frac{b^2}{2} + 2(a+\frac{b}{2})^2 M^2 + 2b^2} =$$

$$= \sqrt{2(a+\frac{b}{2})^2 (1+M^2) + \frac{5}{2} b^2}$$

$$2a^2 + 2ab + \frac{b^2}{2}$$

$$= \frac{4a^2 + \overset{2 \cdot 2ab}{4ab} + b^2}{2} \leq \frac{4a^2 + 4a^2 + b^2 + b^2}{2} = 4a^2 + b^2$$

$$\overset{!}{2xy \leq x^2 + y^2}$$

ili

$$(a+\frac{b}{2})^2 \leq$$

$$\leq 2\left(a^2 + \frac{b^2}{4}\right)$$

$$= 2a^2 + \frac{b^2}{2}$$

$$\leq \sqrt{4(1+M^2)a^2 + (\frac{7}{2} + M^2) b^2}$$

K

$\leq$

$\alpha = \max\{4(1+M^2), \frac{7}{2}+M^2\}$

$\overset{(*)}{\leq} \sqrt{\alpha(a^2+b^2)} = \sqrt{\alpha}\sqrt{a^2+b^2}$

$\sqrt{a^2+b^2}$

$\mathbin{/\mkern-6mu/}$

za $\varepsilon > 0$ $\exists \delta = \frac{\varepsilon}{\sqrt{\alpha}}$ t.d. $\|(a,b)-(0,0)\| < \delta$

$\Rightarrow \|(x_1,x_2)(t) - 0\| < \varepsilon$

gdje $x_1(0)=a$
$x_2(0)=b$.

(7)

$x'' + 9x = \sin t$ $(*)$

$\varphi(t) = \frac{1}{8}\sin t$, $t \geq 0$

$x = y + \frac{1}{8}\sin t \to$ translacija

$x'' = y'' - \frac{1}{8}\sin t$

Uvrstimo u $(*)$:

$y'' - \cancel{\frac{1}{8}\sin t} + 9y + \cancel{\frac{9}{8}\sin t = \sin t}$

$\boxed{\begin{cases} y'' + 9y = 0. \\ y = 0 \end{cases}}$

$\lambda^2 + 9 = 0$ $\lambda = \pm 3i$.

$y = C_1 \cos(3t) + C_2 \sin(3t)$

$\boxed{\text{za } \varphi = 0 \quad \begin{matrix} \varphi(0) = 0 \\ \varphi'(0) = 0 \end{matrix}}$

$\left.\begin{matrix} y(0) = a \\ y'(0) = b \end{matrix}\right\} \begin{matrix} C_1 = a \\ 3C_2 = b \Rightarrow C_2 = \frac{b}{3}. \end{matrix}$

$|y(t) - 0| = \left|a\cos 3t + \frac{b}{3}\sin 3t\right| \leq |a| + \left|\frac{b}{3}\right| \leq |a| + |b|$

$\|(a,b)-(0,0)\| < \delta$

$\mathbin{/\mkern-6mu/}$

$\leq \sqrt{2}\sqrt{a^2+b^2} < \varepsilon$

za $\varepsilon > 0$ $\exists \delta = \frac{\varepsilon}{\sqrt{2}}$ t.d. iz $\sqrt{a^2+b^2} < \delta$

$\Rightarrow |y(t) - 0| < \varepsilon$